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From very-extended to overextended gravity and M-theories

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Abstract: The formulation of gravity and M-theories as very-extended Kac-Moody invariant theories encompasses, for each very-extended algebra G+++, two distinct actions invariant under the overextended Kac-Moody subalgebra G++. The first carries a euclidean signature and is the generalisation to G++ of the E10-invariant action proposed in the context of M-theory and cosmological billiards. The second action carries various Lorentzian signatures revealed through various equivalent formulations related by Weyl transformations of fields. It admits exact solutions, identical to those of the maximally oxidised field theories and of their exotic counterparts, which describe intersecting extremal branes smeared in all directions but one. The Weyl transformations of G++ relates these solutions by conventional and exotic dualities. These exact solutions, common to the Kac-Moody theories and to space-time covariant theories, provide a laboratory for analysing the significance of the infinite set of fields appearing in the Kac-Moody formulations.

Keywords: M-Theory, String Duality, Space-Time Symmetries, Classical Theories of Gravity.

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Contents

1. Introduction

A maximally oxidised theory is a lagrangian theory of gravity coupled to forms and dilatons defined in the highest possible space-time dimension D which upon dimensional reduction to three dimensions exhibits the symmetry of a simple Lie group $\mathcal{G}$ realised on a coset space $\mathcal{G}/\mathcal{H}$. Here $\mathcal{H}$ is the maximal compact subgroup of $\mathcal{G}$. The maximally oxidised actions have been constructed for all $\mathcal{G}$ [1] in the split case considered here. They comprise in particular pure gravity in D dimensions and the low energy effective actions of the bosonic string and of the bosonic sector of M-theory.

It has been conjectured that these theories, or some extensions of them, possess the much larger very-extended Kac-Moody symmetry $\mathcal{G}^{+++}$. $\mathcal{G}^{+++}$ algebras are defined by the Dynkin diagrams depicted in figure 1, obtained from those of $\mathcal{G}$ by adding three nodes [2]. One first adds the affine node, labelled 3 in the figure, then a second node, 2, connected to it by a single line and defining the overextended $\mathcal{G}^{++}$ algebra, then a third one, 1, connected by a single line to the overextended node. Such $\mathcal{G}^{+++}$ symmetries were first conjectured in the aforementioned particular cases [3, 4] and the extension to all $\mathcal{G}^{+++}$ was proposed in [5]. In a different development, the study of the properties of cosmological solutions in the vicinity of a space-like singularity, known as cosmological billiards [6], revealed an overextended symmetry $\mathcal{G}^{++}$ for all maximally oxidised theories [7, 8].

To explore the possible fundamental significance of these huge symmetries two lagrangian formulations [9, 10] *explicitly* invariant under such infinite-dimensional Kac-Moody algebras have been proposed.

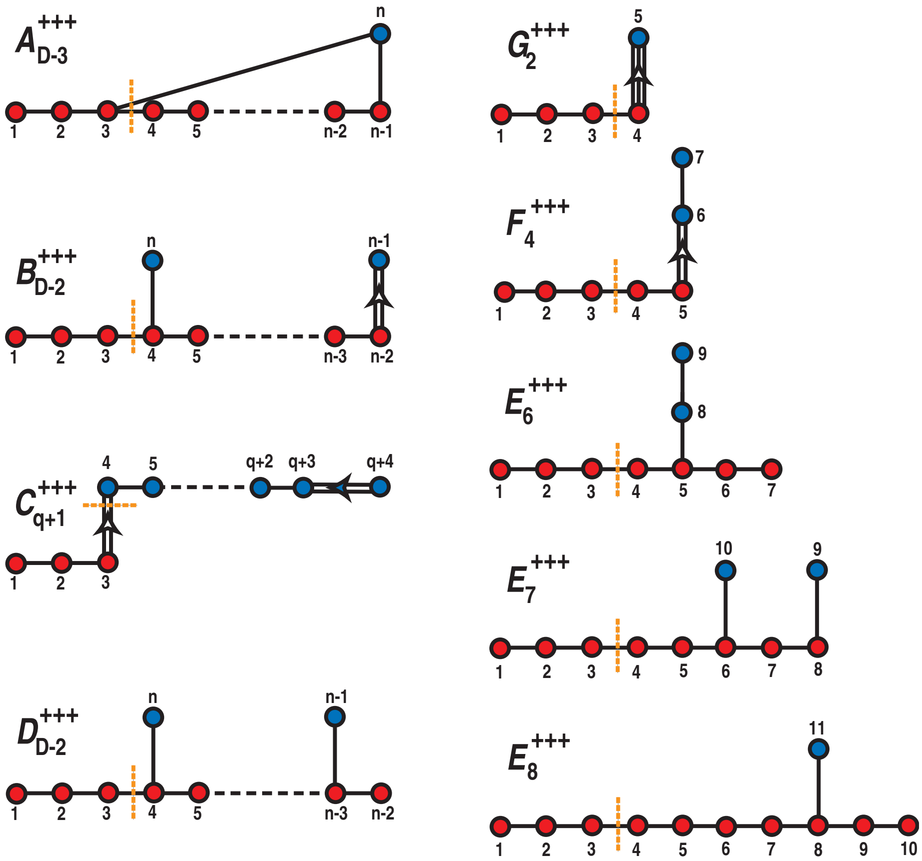

Figure 1: The nodes labelled 1,2,3 define the Kac-Moody extensions of the Lie algebras. The horizontal line starting at 1 defines the 'gravity line', which is the Dynkin diagram of a A_{D-1} subalgebra. Note that for some Kac-Moody algebras the choice of a gravity line is not unique, in which case we take here the one shown in the figure.

First [9] an $E_{10} \equiv E_8^{++}$-invariant action was constructed in the context of M-theory as a reparametrisation invariant σ-model of fields depending on one parameter t and living on the coset space E_8^{++}/K_8^{++}. Here K_8^{++} is the subalgebra of E_8^{++} invariant under the Chevalley involution. The action is built in a recursive way by a level expansion of E_8^{++} with respect to its subalgebra A_9 whose Dynkin diagram is the 'gravity line' defined in figure 1, with the node 1 deleted. The level of an irreducible representation of A_9 occurring in the decomposition of the adjoint representation of E_8^{++} counts the number of times the simple root not pertaining to the gravity line appears in the decomposition. The σ-model, limited to the real roots up to level 3, reveals a perfect match with the bosonic equations of motion of 11-dimensional supergravity in the vicinity of the space-like singularity of the E_8^{++} cosmological billiards, where the fields depend only on time.[1] In the dictionary relating the σ-model to supergravity, the parameter t is identified with time. It was conjectured that space derivatives are hidden in some higher level fields of the σ-model. This approach

[1] See also [11] for an analysis in a different formulation, and [12] for a discussion of E_8^{++} adapted to IIB supergravity.

is straightforwardly generalised, as seen below, to actions $S_{\mathcal{G}_C^{++}}$ related to cosmological billiards and invariant under the overextended $\mathcal{G}^{++}$. Such $\mathcal{G}^{++}$-invariant formulation singles out naturally a time coordinate.

The second approach [10] yields a $\mathcal{G}^{+++}$-invariant action $S_{\mathcal{G}^{+++}}$ which puts all the space-time coordinates on the same footing. This is achieved by a reparametrisation invariant σ-model with fields depending on a parameter ξ spanning a world-line a priori unrelated to space-time. One uses a level decomposition of $\mathcal{G}^{+++}$ with respect to the subalgebra A_{D-1} of its gravity line[2] and D is identified to the space-time dimension. The ξ-dependent fields live in a coset space $\mathcal{G}^{+++}/K^{+++}$ where the subalgebra K^{+++} is invariant under a temporal involution [10] which preserve the Lorentz algebra $\mathrm{SO}(1, D-1)$ and ensures that the action $S_{\mathcal{G}^{+++}}$ is Lorentz invariant at each level.

In this paper, we analyse the $\mathcal{G}^{++}$ content of the $\mathcal{G}^{+++}$-invariant action $S_{\mathcal{G}^{+++}}$. We find that the $\mathcal{G}^{++}$-invariant action $S_{\mathcal{G}_C^{++}}$ is obtained by putting to zero in $S_{\mathcal{G}^{+++}}$, consistently with all its equations of motion, all the fields which do not appear in $S_{\mathcal{G}_C^{++}}$. This consistent truncation of the $\mathcal{G}^{+++}$ theory implies that all the solutions of the equations of motion of $S_{\mathcal{G}_C^{++}}$ are also solutions of the equations of motion of $S_{\mathcal{G}^{+++}}$. We then find that $S_{\mathcal{G}^{+++}}$ contains another action $S_{\mathcal{G}_B^{++}}$ obtained by performing the consistent truncation *after* conjugation by a Weyl reflection in $\mathcal{G}^{+++}$. The action $S_{\mathcal{G}_B^{++}}$ is also invariant under a $\mathcal{G}^{++}$ algebra but is nonequivalent to $S_{\mathcal{G}_C^{++}}$ and singles out naturally a space direction. Through Weyl transformations in $\mathcal{G}^{++}$ expressed in terms of fields, $S_{\mathcal{G}_B^{++}}$ can be formulated in various equivalent forms endowed with various lorentzian signatures $(-\ldots-,+\ldots+)$, a consequence of the non-commutativity of the temporal involution with Weyl reflections [17, 18]. Exact solutions of $S_{\mathcal{G}_B^{++}}$ are obtained in accordance with references [10, 19]. These solutions are *identical* to those of the covariant Einstein and field equations describing, depending on the signature, conventional or exotic intersecting extremal branes smeared in all directions but one [20, 21]. They transform into each other by 'duality' Weyl transformations. As all these brane solutions of $S_{\mathcal{G}_B^{++}}$ are also solutions of the $\mathcal{G}^{+++}$-invariant action $S_{\mathcal{G}^{+++}}$, the latter necessarily contains solutions of exotic counterparts of the original maximally oxidised theories. In particular the $E_{11} \equiv E_8^{+++}$-invariant action contains in addition to the M-theory solutions, the exotic branes of the related M' and M* theories [21]–[24]. Of course $S_{\mathcal{G}^{+++}}$ can, as $S_{\mathcal{G}_B^{++}}$, be formulated with different signatures related by field transformations.

The fact that $S_{\mathcal{G}_B^{++}}$, and hence $S_{\mathcal{G}^{+++}}$, contains *exact* intersecting extremal brane solutions of space-time covariant theories compactified to all dimensions but one provides a laboratory to analyse the significance of at least some subset of the infinite many fields appearing in the Kac-Moody invariant theories. Extremal branes in more non-compact dimensions differ from the one dimensional ones only by the dependance of a harmonic function on the number of non-compact dimensions. For such decompactified solutions to exist in the Kac-Moody theory, higher level fields must provide the derivatives [9, 10, 14, 25]

[2] Level expansions of very-extended algebras in terms of the subalgebra A_{D-1} have been considered in [13, 14, 15]. The importance of the gravity line was recognised long ago by B. Julia [16].

needed to obtain higher dimensional harmonic functions. This test is crucial to settle the issue of whether or not the Kac-Moody theories discussed here can really describe uncompactified space-time covariant theories.

The paper is organised as follows. In section 2, we recall the construction of the $\mathcal{G}^{+++}$-invariant actions $S_{\mathcal{G}^{+++}}$. In section 3, we show how the 'cosmological' $\mathcal{G}^{++}$-invariant action $S_{\mathcal{G}_C^{++}}$ is obtained from $S_{\mathcal{G}^{+++}}$ by a consistent truncation. In section 4, we obtain the 'brane' action $S_{\mathcal{G}_B^{++}}$ which arises from a Weyl transformation in $\mathcal{G}^{+++}$ and a consistent truncation. We show how to formulate $S_{\mathcal{G}_B^{++}}$ in various equivalent ways to exhibit various lorentzian signatures. Differential relations for the Weyl transformations of fields ensuring the equivalence of the different formulations are obtained. The equivalence proves that, in addition to conventional ones, exotic dualities are present in $\mathcal{G}^{+++}$. This is made explicit in the particular case of E_{11} and is in agreement with the results of reference [17, 18], linking in $\mathcal{G}^{+++}$ M-theory to M' and M*-theories. In section 5, we derive the exact solutions of $S_{\mathcal{G}_B^{++}}$, and hence of $S_{\mathcal{G}^{+++}}$, which are identical to the space-time solutions describing intersecting extremal branes, conventional and exotic, smeared in all directions but one. We illustrate by a specific non-trivial example how dualities transforming these solutions into one another operate. In section 6 we stress perspectives suggested by our results.

2. $\mathcal{G}^{+++}$-invariant action

Actions $S_{\mathcal{G}^{+++}}$ invariant under non-linear transformations of $\mathcal{G}^{+++}$ are constructed recursively from a level decomposition with respect to a subalgebra A_{D-1} where D is interpreted as the space-time dimension. The action is defined in a reparametrisation invariant way on a world-line, a priori unrelated to space-time, in terms of fields $\varphi(\xi)$ where ξ spans the world-line. The fields $\varphi(\xi)$ live in a coset space $\mathcal{G}^{+++}/K^{+++}$ where the subalgebra K^{+++} is invariant under a 'temporal involution' preserving at each level a Lorentz algebra[3] $\mathrm{SO}(1, D-1) = A_{D-1} \cap K^{+++}$.

We now recall in more detail the construction of these $\mathcal{G}^{+++}$-invariant theories [10].

$\mathcal{G}^{+++}$ contains a subalgebra GL(D) such that $\mathrm{SL}(D)(= A_{D-1}) \subset \mathrm{GL}(D) \subset \mathcal{G}^{+++}$. The generators of the GL(D) subalgebra are taken to be $K^a{}_b$ $(a, b = 1, 2, \ldots, D)$ with commutation relations

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b \,. \tag{2.1}$$

The $K^a{}_b$ along with the abelian generator R, which is present when the corresponding maximally oxidised action $S_{\mathcal{G}}$ has one dilaton[4], are the level zero generators. The step operators of level greater than zero are tensors $R_{d_1 \ldots d_s}{}^{c_1 \ldots c_r}$ of the A_{D-1} subalgebra. The lowest levels contain antisymmetric tensor step operators $R^{a_1 a_2 \ldots a_r}$ associated with electric and magnetic roots arising from the dimensional reduction of field strength forms in $S_{\mathcal{G}}$.

[3]In section 4, we shall see that G^{+++} contains other A_{D-1} subalgebras intersecting differently with K^{+++}.

[4]All the maximally oxidised theories have at most one dilaton except the C_{q+1}-series. The maximally oxidised theory C_{q+1} is a four dimensional theory containing q dilatons and C_{q+1}^{+++} is constructed with q abelian generators R_i $i = 1 \ldots q$ (see for instance [10, appendix A3]). To avoid crowding of indices, we consider here only the theories with at most one dilaton.

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They satisfy the tensor and scaling relations

$$[K^a{}_b, R^{a_1\dots a_r}] = \delta_b^{a_1} R^{aa_2\dots a_r} + \dots + \delta_b^{a_r} R^{a_1\dots a_{r-1}a}\,, \tag{2.2}$$

$$[R, R^{a_1\dots a_r}] = -\frac{\varepsilon_A a_A}{2} R^{a_1\dots a_r}\,, \tag{2.3}$$

where a_A is the dilaton coupling constant to the field strength form and ε_A is $+1\,(-1)$ for an electric (magnetic) root [5]. The generators obey the invariant scalar product relations

$$\langle K^a{}_a K^b{}_b\rangle = G_{ab}\,, \qquad \langle K^b{}_a K^d{}_c\rangle = \delta^b_c \delta^d_a \; a \neq b\,, \qquad \langle RR\rangle = \frac{1}{2}\,, \tag{2.4}$$

$$\langle R_{b_1\dots b_s}{}^{a_1\dots a_r}, \bar{R}_{d_1\dots d_r}{}^{c_1\dots c_s}\rangle = \delta^{c_1}_{b_1}\dots\delta^{c_s}_{b_s}\delta^{a_1}_{d_1}\dots\delta^{a_r}_{d_r}\,. \tag{2.5}$$

Here $G = I_D - \frac{1}{2}\Xi_D$ where Ξ_D is a D-dimensional matrix with all entries equal to unity and $\bar{R}_{d_1\dots d_r}{}^{c_1\dots c_s}$ designates the negative step operator conjugate to $R_{c_1\dots c_s}{}^{d_1\dots d_r}$.

The temporal involution Ω_1 generalises the Chevalley involution to allow identification of the index 1 to a time coordinate in SO$(1, D-1)$. It is defined by

$$K^a{}_b \overset{\Omega_1}{\mapsto} -\epsilon_a\epsilon_b K^b{}_a \qquad R \overset{\Omega_1}{\mapsto} -R \qquad , \quad R_{d_1\dots d_s}{}^{c_1\dots c_r} \overset{\Omega_1}{\mapsto} -\epsilon_{c_1}\dots\epsilon_{c_r}\epsilon_{d_1}\dots\epsilon_{d_s}\bar{R}_{c_1\dots c_r}{}^{d_1\dots d_s}\,, \tag{2.6}$$

with $\epsilon_a = -1$ if $a = 1$ and $\epsilon_a = +1$ otherwise. It leaves invariant a subalgebra K^{+++} of $\mathcal{G}^{+++}$.

The fields $\varphi(\xi)$ living in the coset space $\mathcal{G}^{+++}/K^{+++}$ parametrise the Borel group built out of Cartan and positive step operators in $\mathcal{G}^{+++}$. Its elements $\mathcal{V}$ may formally be written as

$$\mathcal{V}(\xi) = \exp\left(\sum_{a\geq b} h_b{}^a(\xi) K^b{}_a - \phi(\xi) R\right) \exp\left(\sum \frac{1}{r!s!} A_{b_1\dots b_s}{}^{a_1\dots a_r}(\xi) R_{a_1\dots a_r}{}^{b_1\dots b_s} + \cdots\right), \tag{2.7}$$

where the first exponential contains only level zero operators and the second one the positive step operators of levels strictly greater than zero. As is customary in the field, we proceed formally as in the finite dimensional case. Our purpose is to shed light on the symmetry structure, not to deal with the intricate aspects of analysis. Furthermore, note that when the denominator is not compact, the exponential representation might not cover the entire coset space. This does not affect the considerations of this paper.

Defining

$$dv(\xi) = d\mathcal{V}\mathcal{V}^{-1} \qquad d\tilde{v}(\xi) = -\Omega_1\, dv(\xi) \qquad dv_{\rm sym} = \frac{1}{2}(dv + d\tilde{v})\,, \tag{2.8}$$

one obtains, in terms of the ξ-dependent fields, an action $S_{\mathcal{G}^{+++}}$ invariant under global $\mathcal{G}^{+++}$ transformations, defined on the coset $\mathcal{G}^{+++}/K^{+++}$

$$S_{\mathcal{G}^{+++}} = \int d\xi \frac{1}{n(\xi)} \left\langle \left(\frac{dv_{\rm sym}(\xi)}{d\xi}\right)^2 \right\rangle, \tag{2.9}$$

where $n(\xi)$ is an arbitrary lapse function ensuring reparametrisation invariance on the world-line. One has

$$dv_{\rm sym} = dv^0_{\rm sym} + \sum_A dv^{(A)}_{\rm sym}\,, \tag{2.10}$$

where $dv^0_{\rm sym}$ contains all the level zero contributions. One gets

$$dv^0_{\rm sym} = -\frac{1}{2}\sum_{a\geq b}[e^h(de^{-h})]_b{}^a(K^b{}_a - \Omega_1 K^b{}_a) - d\phi R\,, \tag{2.11}$$

$$dv^{(A)}_{\rm sym} = \frac{1}{2r!s!} DA_{\mu_1\ldots\mu_r}{}^{\nu_1\ldots\nu_s}\,\exp(-\lambda\phi)\,e_{a_1}^{\mu_1}\ldots e_{a_r}^{\mu_r} e_{\nu_1}^{b_1}\ldots e_{\nu_s}^{b_s}\left(R_{b_1\ldots b_s}{}^{a_1\ldots a_r} - \Omega_1 R_{b_1\ldots b_s}{}^{a_1\ldots a_r}\right), \tag{2.12}$$

where $e_\mu{}^a = (e^{-h(\xi)})_\mu{}^a$, λ is the generalisation of the scale parameter $-\varepsilon_A a_A/2$ to all roots and $D/D\xi$ is a covariant derivative generalising $d/d\xi$ through non-linear terms arising from non-vanishing commutators between positive step operators.

Writing

$$S_{\mathcal{G}^{+++}} = S^{(0)}_{\mathcal{G}^{+++}} + \sum_A S^{(A)}_{\mathcal{G}^{+++}}\,, \tag{2.13}$$

where $S^{(0)}_{\mathcal{G}^{+++}}$ contains all level zero contributions, one obtains

$$S^{(0)}_{\mathcal{G}^{+++}} = \frac{1}{2}\int d\xi \frac{1}{n(\xi)}\left[\frac{1}{2}\left(g^{\mu\nu}g^{\sigma\tau} - \frac{1}{2}g^{\mu\sigma}g^{\nu\tau}\right)\frac{dg_{\mu\sigma}}{d\xi}\frac{dg_{\nu\tau}}{d\xi} + \frac{d\phi}{d\xi}\frac{d\phi}{d\xi}\right], \tag{2.14}$$

$$S^{(A)}_{\mathcal{G}^{+++}} = \frac{1}{2r!s!}\int d\xi \frac{e^{-2\lambda\phi}}{n(\xi)}\left[\frac{DA_{\mu_1\ldots\mu_r}{}^{\nu_1\ldots\nu_s}}{d\xi} g^{\mu_1\mu'_1}\cdots g^{\mu_r\mu'_r} g_{\nu_1\nu'_1}\cdots g_{\nu_s\nu'_s}\frac{DA_{\mu'_1\ldots\mu'_r}{}^{\nu'_1\ldots\nu'_s}}{d\xi}\right]. \tag{2.15}$$

The ξ-dependent fields $g_{\mu\nu}$ are defined as $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$. The appearance of the Lorentz metric η_{ab} with $\eta_{11} = -1$ is a consequence of the temporal involution Ω_1. The metric $g_{\mu\nu}$ allows a switch from the Lorentz indices (a, b) of the fields appearing in eq. (2.7) to GL(D) indices (μ, ν).

Exact solutions have been found for all $\mathcal{G}^{+++}$ theories. In [10] solutions describing the algebraic properties of the BPS extremal branes have been discovered. Each single extremal p-brane is characterised by *one* non-zero field multiplying a component of an antisymmetric tensor step operator of low level. Each extremal brane is related in this way to a *real positive root* in $\mathcal{G}^{+++}$. In [19] these results have been extended and exact solutions of $S_{\mathcal{G}^{+++}}$ for all the extremal intersecting brane solutions [20, 21] of the maximally oxidised theory were found. The intersection rules determining such configurations are neatly encoded in the $\mathcal{G}^{+++}$ algebra. They are expressed as orthogonality conditions on the real roots characterising the intersecting branes.

3. From $\mathcal{G}^{+++}$ to the cosmological $\mathcal{G}^{++}_C$-invariant action

Consider the overextended algebra $\mathcal{G}^{++}_C$ obtained from the very-extended algebra $\mathcal{G}^{+++}$ by deleting the node labelled 1 from the Dynkin diagrams of $\mathcal{G}^{+++}$ depicted in figure 1. This algebra is realised in the space of Kasner solutions of the action $S_{\mathcal{G}^{+++}}$ and more generally in the cosmological billiards solutions.[5] In this section we show how an action

[5]In what follows, we use the notation $\mathcal{G}^{++}$ when referring to the multiplication table of the overextended algebra and add an index to $\mathcal{G}^{++}$ when we want to keep track of the embedding in $\mathcal{G}^{+++}$.

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invariant under $\mathcal{G}_C^{++}$ emerges from the $\mathcal{G}^{+++}$-invariant action $S_{\mathcal{G}^{+++}}$ eq. (2.13) through a *consistent* truncation. This 'cosmological' action $S_{\mathcal{G}_C^{++}}$ generalises to all $\mathcal{G}^{++}$ the E_{10} ($\equiv E_8^{++}$) action of reference [9]. It is obtained below from $S_{\mathcal{G}^{+++}}$ by putting to zero in the coset representative eq. (2.7) the field multiplying the Chevalley generator $H_1 = K^1{}_1 - K^2{}_2$ and all the fields multiplying the positive step operators associated to roots whose decomposition in terms of simple roots contains the deleted root α_1. Such a truncation of the action $S_{\mathcal{G}^{+++}}$ will be shown to be consistent with all its equations of motion.

Consider first the fields $A_{b_1\ldots b_s}{}^{a_1\ldots a_r}(\xi)$ multiplying the positive step operators $R_{a_1\ldots a_r}{}^{b_1\ldots b_s}$. The $A_{b_1\ldots b_s}{}^{a_1\ldots a_r}(\xi)$ equated to zero fall into two classes. First the set of all fields appearing in irreducible representations of A_{D-1} whose lowest weight contains α_1. Second, in the other representations of A_{D-1}, the set of tensor components with at least one time index. The consistency of the truncation of the fields from the first class is obvious as all terms in the action eq. (2.13) either contain no such field or contain at least two of them. Fields of the second class may occur linearly but are then multiplied by non diagonal metrics $g_{1\hat{\mu}}$ where the hatted indices $\hat{\mu}$ run from 2 to D. In the triangular gauge for $h_\mu{}^a$ stemming from the Borel group defined in eq. (2.7), the linear terms in the second class A-fields are necessarily multiplied by at least one $h_1{}^a$-field ($a > 1$) associated through $K^1{}_a$ to a root α_1. Taking

$$g_{1\hat{\mu}} = 0\,, \tag{3.1}$$

we ensure the consistency of putting to zero the fields of the second class and in fact of all the fields multiplying step operators associated to a root α_1, including those of level zero. Indeed, the equation of motions of the second class A-fields are now satisfied. Consider now the equations of motion of the $g_{1\hat{\mu}}$. All the terms containing $g_{1\hat{\mu}}$ contain necessarily at least one other $g_{1\hat{\nu}}$ in eq. (2.14) or at least one A-field component put to zero in eq. (2.15). The equations of motion of the $g_{1\hat{\mu}}$ are thus also consistent with the truncation.

We now turn to the fields multiplying Cartan generators. Labelling q_1 the field multiplying H_1 in the Chevalley basis we express the equation $q_1 = 0$ in terms of the metric $g_{\mu\nu}$. We define

$$p_a = -h_a{}^a\,, \tag{3.2}$$

where $h_a{}^a$ is the coefficient of $K^a{}_a$ in eq. (2.7) . One gets for all $\mathcal{G}^{+++}$ [5] the equivalence

$$q_1 = 0 \qquad \Longleftrightarrow \qquad p_1 = \sum_{a=2}^{D} p_a\,. \tag{3.3}$$

Using the fact that the coset model eq. (2.9) was computed in the triangular Borel gauge and using eq. (3.1), we rewrite eq. (3.3) as

$$g_{11} = \mathbf{g}\,, \tag{3.4}$$

$$\mathbf{g} \equiv \det g_{\hat{\mu}\hat{\nu}}\,. \tag{3.5}$$

Consistency of the truncation from $\mathcal{G}^{+++}$ to $\mathcal{G}^{++}$ requires that eq. (3.4) satisfy the equation of motion of g_{11}. Only variations with respect to g_{11} in the action eq. (2.14) have to be considered, as the variations in eq. (2.15) are automatically satisfied because

g_{11} multiplies A-fields already equated to zero. Using eq. (3.1) we get

$$\frac{d^2}{d\xi^2}\ln|g_{11}| - \frac{d^2}{d\xi^2}\ln \mathbf{g} = 0\,, \tag{3.6}$$

which admits as solution eq. (3.4).

To obtain the actions $S_{\mathcal{G}_C^{++}}$ whose equations of motion are all contained in the equations of motion obtained from $S_{\mathcal{G}^{+++}}$ given by eq. (2.9), it thus suffices to substitute in eq. (2.14) g_{11} by its value given in eq. (3.4) and equate to zero in eq. (2.15) all first and second class A-fields. Denoting the remaining A-fields by B and using the hatted indices $\hat{\mu}, (\mu = 2, \dots D)$, we obtain

$$S_{\mathcal{G}_C^{++}} = S^{(0)}_{\mathcal{G}_C^{++}} + \sum_B S^{(B)}_{\mathcal{G}_C^{++}}\,, \tag{3.7}$$

where

$$S^{(0)}_{\mathcal{G}_C^{++}} = \frac{1}{2}\int dt \frac{1}{n(t)}\left[\frac{1}{2}(g^{\hat{\mu}\hat{\nu}}g^{\hat{\sigma}\hat{\tau}} - g^{\hat{\mu}\hat{\sigma}}g^{\hat{\nu}\hat{\tau}})\frac{dg_{\hat{\mu}\hat{\sigma}}}{d\tau}\frac{dg_{\hat{\nu}\hat{t}}}{dt} + \frac{d\phi}{dt}\frac{d\phi}{dt}\right], \tag{3.8}$$

$$S^{(B)}_{\mathcal{G}_C^{++}} = \frac{1}{2r!s!}\int dt \frac{e^{-2\lambda\phi}}{n(t)}\left[\frac{DB_{\hat{\mu}_1\dots\hat{\mu}_r}{}^{\hat{\nu}_1\dots\hat{\nu}_s}}{dt} g^{\hat{\mu}_1\hat{\mu}'_1}\cdots g^{\hat{\mu}_r\hat{\mu}'_r} g_{\hat{\nu}_1\hat{\nu}'_1}\cdots\ , g_{\hat{\nu}_s\hat{\nu}'_s}\frac{DB_{\hat{\mu}'_1\dots\hat{\mu}'_r}{}^{\hat{\nu}'_1\dots\hat{\nu}'_s}}{dt}\right]. \tag{3.9}$$

Here we renamed ξ as t. Eqs. (3.7)–(3.9) follow from Eq.(2.9) by replacing $dv_{\rm sym}$ by $dv_{{\rm sym}(+)}$, which describes a motion on the coset $\mathcal{G}^{++}/K^{++}_{(+)}$ where $K^{++}_{(+)}$ is the subalgebra of $\mathcal{G}^{++}$ invariant under the Chevalley involution. In particular, we see that for $\mathcal{G}^{++} = E_8^{++}$, we recover as a consistent truncation of $S_{E_8^{+++}}$ given by eq. (2.9) the action for the $E_{10} \equiv E_8^{++}$-invariant theory proposed in reference [9].

The fact that the solutions of the $\mathcal{G}^{++}$-equations of motion are also solutions of the $\mathcal{G}^{+++}$-equations of motion corresponding to a particular choice of the initial conditions can be viewed in a different way. Consider the general solution of the $\mathcal{G}^{+++}/K^{+++}$ non linear sigma-model. In the notations of [6], it reads

$$\mathcal{M}(\xi) = \mathcal{M}(0)\cdot \exp(\xi J)\,, \tag{3.10}$$

where $\mathcal{M}$ is defined as $\mathcal{M} = \mathcal{V}^{T'}\mathcal{V}$ in terms of the $\mathcal{G}^{+++}$-group element $\mathcal{V}$. Here, the generalised transposition is defined in terms of the involution Ω_1 as $E^{T'} = -\Omega_1(E)$ for any Lie algebra element E (and extended by the rule $(AB)^{T'} = B^{T'}A^{T'}$). In (3.10), the J's are the conserved currents and belong to $\mathcal{G}^{+++}$. If we choose the initial conditions in such a way that $\mathcal{M}(0)$ belongs to the $\mathcal{G}^{++}$-subgroup and the current J is in the subalgebra $\mathcal{G}^{++}$, then we see from (3.10) that $\mathcal{M}(\xi)$ is in the $\mathcal{G}^{++}$-subgroup for all "times" ξ. This proves consistency of the truncation at the level of the equations of motion.

4. From $\mathcal{G}^{+++}$ to the brane $\mathcal{G}_B^{++}$-invariant action

4.1 Weyl reflections of the gravity line

The Weyl reflection W_{α_1} in the hyperplane perpendicular to α_1 generates a subalgebra $\mathcal{G}_2^{++}$ conjugate to $\mathcal{G}_C^{++}$ in $\mathcal{G}^{+++}$. More generally, performing Weyl reflections from roots of the gravity line, we generate in this way $(D-1)$ subalgebras $\mathcal{G}_a^{++}, (a = 2, \dots, D)$, which are

gravity line	$K^1{}_2$	$K^2{}_3$	$K^3{}_4$	$\cdots$	$K^{D-1}{}_D$	time coordinate
Ω	$+$	$-$	$-$	$-$	$-$	1
Ω'	$+$	$+$	$-$	$-$	$-$	2

Table 1: Involution switch from Ω to Ω' due to the Weyl reflection W_{α_1}.

all conjugate in $\mathcal{G}^{++}$. While the generators of $\mathcal{G}_C^{++}$ were labelled by spatial indices only, the index a in $\mathcal{G}_a^{++}$ must be interpreted as a time coordinate. The fact that we have $D-1$ different identifications of the time coordinate follows from the non-commutativity of the time involution with the Weyl reflections [17] as seen below. This non-commutativity will be exploited in the more general context of section 4.2 but is well illustrated by its effect on the gravity line.

Expressing a Weyl transformation W as a conjugation by a group element U_W of $\mathcal{G}^{+++}$, we define the involution Ω' operating on the conjugate elements by

$$U_W\,\Omega T\,U_W^{-1} = \Omega'\,U_W T U_W^{-1}\,, \tag{4.1}$$

where T is any generator of $\mathcal{G}^{+++}$. Applying eq. (4.1) to the Weyl reflection W_{α_1} generates the subalgebra $\mathcal{G}_2^{++}$ conjugate to $\mathcal{G}_C^{++}$. One gets

$$\begin{aligned} U_1\,\Omega K^2{}_1\,U_1^{-1} &= \rho K^2{}_1 = \rho\Omega' K^1{}_2\,, \\ U_1\,\Omega K^1{}_3\,U_1^{-1} &= \sigma K^3{}_2 = \sigma\Omega' K^2{}_3\,, \\ U_1\,\Omega K^i{}_{i+1}\,U_1^{-1} &= -\tau K^{i+1}_i = \tau\Omega' K^i{}_{i+1} \quad i>2\,. \end{aligned} \tag{4.2}$$

Here ρ,σ,τ are plus or minus signs which may arise as step operators are representations of the Weyl group up to signs. Eq. (4.2) illustrate the general result that such signs always cancel in the determination of Ω' because they are identical in the Weyl transform of corresponding positive and negative roots, as their commutator is in the Cartan subalgebra which forms a true representation of the Weyl group. The content of eq. (4.2) is represented in table 1. The signs below the generators of the gravity line indicate the sign in front of the negative step operator obtained by the involution: a minus sign is in agreement with the conventional Chevalley involution and indicates that the indices in $K^a{}_{a+1}$ are both either space or time indices while a plus sign indicates that one index must be time and the other space.

Table 1 show that the time coordinates in $\mathcal{G}_2^{++}$ must now be identified either with 2, or with all indices $\neq 2$. We choose the first description, which leaves unaffected coordinates attached to planes invariant under the Weyl transformation. Similarly the time coordinate in $\mathcal{G}_a^{++}$ is taken to be a.

One may now apply the analysis of section 3 to obtain actions $S_{\mathcal{G}_a^{++}}$ invariant under $\mathcal{G}_a^{++}$, namely we equate as before to zero all the fields in the $\mathcal{G}^{+++}$-invariant action eq. (2.13) which multiply generators not involving the root α_1, but this truncation is performed *after* the Weyl transformation which transmutes the time index 1 to a space index. The actions $S_{\mathcal{G}_a^{++}}$ are then formally identical to the one given by eqs. (3.7)–(3.9) but with a Lorentz signature for the metric, which in the flat coordinates amounts to a negative sign for the Lorentz metric component η_{aa}, and with ξ identified to the missing space coordinate instead of t.

The $D-1$ actions $S_{\mathcal{G}_a^{++}}$ for $a = 2, \ldots, D$ differ only by the index identifying the time coordinate and the concomitant identification of ξ with a space coordinate. They are thus equal up to a trivial redefinition of all tensor fields by an interchange of indices. This redefinition may be viewed as the Weyl transformations generated by the roots of the subalgebra A_{D-2} on the space of fields. Equivalence under Weyl transformations from roots which do not belong to the gravity line are far less trivial and will now be examined.

4.2 General Weyl reflections

The $D-1$ actions $S_{\mathcal{G}_a^{++}}$ differ by the labelling of the time coordinate and have all the same global signature $(1,9)$. They are related through Weyl transformations of $\mathcal{G}^{++}$ from roots of the gravity line. Consider now the Weyl transformations of $\mathcal{G}^{++}$ generated in addition by the simple roots not belonging to A_{D-2}. These will yield actions with different global signatures. All such actions will be shown to be equal and related by field redefinitions. This equivalence realises in the action formalism the general analysis of Weyl transformations by Keurentjes [17, 18]. It will be related to the existence, in addition to conventional duality symmetries, to exotic dualities, which in the particular case of M-theory, are relating it to M* and M'-theories [22, 23]. To illustrate how Weyl transformation change global signatures, we first consider explicitly the signature changes for $E_8^{++} \equiv E_{10}$.

The Weyl reflection from the root associated to any generator R^{abc} multiplying the three-form potential A_{abc} can always be mapped to the Weyl reflection $W_{\alpha_{11}}$ from the simple root α_{11} by products of Weyl reflections of the gravity line, thereby changing the time coordinate, originally positioned at 1, to any position.

Let us choose the time coordinate to be 9 and consider the Weyl reflection $W_{\alpha_{11}}$. The only generator of the gravity line affected by the Weyl transformation is $K^8{}_9$. One has from eq. (4.1)

$$U_{11}\, \Omega R^{8\,10\,11}\, U_{11}^{-1} = -\rho K^9{}_8 = \rho\Omega' K^8{}_9 \,. \tag{4.3}$$

$K^2{}_3$	$K^3{}_4$	$K^4{}_5$	$K^5{}_6$	$K^6{}_7$	$K^7{}_8$	$K^8{}_9$	$K^9{}_{10}$	$K^{10}{}_{11}$	times	$(t,s,\pm)$
$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	9	$(1,9,+)$
$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	10,11	$(2,8,-)$
$-$	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$-$	7,8	$(2,8,-)$
$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	7,8,9,10,11	$(5,5,+)$
$-$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	2,3,4,5,9	$(5,5,+)$
$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	2,3,4,5,10,11	$(6,4,-)$
$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	2,3,4,5,6,7	$(6,4,-)$
$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$-$	2,3,4,5,6,7,9,10,11	$(9,1,+)$

Table 2: Involution switches from Ω to Ω' due to the Weyl reflection $W_{\alpha_{11}}$.

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Imposing that the coordinate 2 remains unaffected by the transformation, we see that both 10 and 11 become time coordinates, as illustrated in table 2. The transformation of the generator $R^{9\,10\,11}$ yields

$$U_{11}\,\Omega R_{9\,10\,11}\,U_{11}^{-1} = \sigma R_{9\,10\,11} = \sigma\Omega' R^{9\,10\,11}\,. \tag{4.4}$$

The action of the involution Ω' on the simple root not pertaining to the gravity line, $\Omega' R^{9\,10\,11} = R_{9\,10\,11}$, differs by a sign from the action of a temporal involution defined according to eq. (2.6) when the times are 10 and 11. This shift of sign is shown in the last column of table 2. It will lead to negative kinetic energy terms in the corresponding actions $S^{(2,8,-)}_{\mathcal{G}^{++}_{(10\,11)}}$ below. Table 2 shows how by repeated use of eqs. (4.3) and (4.4) one reaches the signatures $(5,5,+)$, $(6,4,-)$ and $(9,1,+)$.

It is interesting to illustrate the string interpretation of the Weyl transformations of table 2 in terms of signature changing dualities discussed in [22, 23]. Consider the type IIA interpretation of E_8^{+++}. In the Dynkin diagram of E_8^{+++} in figure 1 the node labelled 10 is no longer on the gravity line and should be redrawn as perpendicular to it at the node labelled 9. Recall that in the embedding of E_8^{++} in E_8^{+++} the coordinate 1 is a space coordinate and that the string interpretation of the Weyl reflection $W_{\alpha_{11}}$ is a double T-duality in the directions 9 and 10 plus an exchange of these directions [26, 27, 5].

The first Weyl reflection of table 2 maps the E_8^{++} signature $(1,9,+)$ to $(2,8,-)$. To get the corresponding string theories we ignore the direction 11 and take into account the direction 1 which is spacelike. The $(1,9,+)$ signature corresponds to type IIA theory. The action of $W_{\alpha_{11}}$ renders 11 timelike. Consequently the string interpretation of the $(2,8,-)$ signature is a string theory with signature $(1,9)$ and wrong signs for the kinetic terms in the RR sector. This defines the type IIA^* theory consistently with string dualities. Indeed a double T-duality with respect to one time direction, 9, and one space direction, 10, maps type IIA onto type IIA^* [22, 23]. The second Weyl reflection of table 2 maps the E_8^{++} signature $(2,8,-)$ with time directions $7,8$ to $(5,5,+)$ with time directions $7,8,9,10,11$. The string interpretation is obtained by the same argument as above and one gets in this way a mapping of IIA_{8+2} to IIA_{6+4} theories, in the notation of [23], where the index $s+t$ designates a theory with s space and t time directions. This result is in agreement with the string dualities. Indeed a double spacelike T-duality in the direction $9,10$ maps IIA_{8+2} to IIA_{6+4} (see [23, figure 5]). The third Weyl reflection of table 2 maps the E_8^{++} signature $(5,5,+)$ with time directions $2,3,4,5,9$ on $(6,4,-)$ with time directions $2,3,4,5,10,11$. The string interpretation maps accordingly type IIA_{5+5} to type IIA^*_{5+5}. These two theories are indeed related by a double T-duality in the time direction 9 and in the space direction 10. The last Weyl reflection of Table 2 maps the E_8^{++} signature $(6,4,-)$ with time directions $2,3,4,5,6,7$ to $(9,1,+)$ with time directions $2,3,4,5,6,7,9,10,11$, in accordance with the string interpretation where type IIA_{4+6} is mapped to type IIA_{2+8} by double spacelike T-duality.

4.3 Weyl equivalence of different space-time signatures

The action $S_{\mathcal{G}_2^{++}}$ is, as $S_{\mathcal{G}_C^{++}}$, given by eqs. (3.7)–(3.9) but with $\eta_{22} = -1$. It can be expressed in the general form eq. (2.9) by replacing dv_{sym} by $dv_{\text{sym}(-)}$ which describes a motion on the coset $\mathcal{G}^{++}/K^{++}_{(-)}$. $K^{++}_{(-)}$ is the subalgebra of $\mathcal{G}^{++}$ invariant under the time involution Ω_2 defined as in eq. (2.6) with 2 as the time coordinate and restricted to $\mathcal{G}^{++}$. To compute $dv_{\text{sym}_{(-)}}$ one specifies a coset representative of $\mathcal{G}^{++}/K^{++}_{(-)}$ and a level expansion about a gravity line endowed with the involution Ω_2. We shall refer to this embedding of $\mathcal{G}^{++}$ in $\mathcal{G}^{+++}$ by $\mathcal{G}^{++}_B$ to distinguish it from $\mathcal{G}^{++}_C$.

The action $S_{\mathcal{G}_2^{++}}$ is characterised by a global signature $(1, 9, +)$ and time coordinate 2. We may reparametrise the coset space $\mathcal{G}^{++}/K^{++}_{(-)}$ by subjecting its elements to a Weyl conjugation by an element U of $\mathcal{G}^{++}$. This selects a new gravity line in $\mathcal{G}^{++}$ which may be endowed with a different involution than Ω_2, as the temporal involution does not in general commute with Weyl transformations. The Borel group and the level expansion accommodating the new involution will then not coincide with the old one, although the new ξ-dependant fields could always in principle be expressed in terms of the old ones. One can construct in this way apparently many distinct actions in terms of these different set of fields. Weyl transformations from the gravity line roots permute the tensor indices and hence may change time indices while Weyl reflections from roots which do not belong to the gravity line may change the global signature, as exemplified for E_8^{++} in table 2. We denote these actions by $S^{(t,s,\varepsilon)}_{\mathcal{G}^{++}_{(i_1 i_2 \ldots i_t)}}$, where the global signature is (t, s, ε) with ε denoting a set of $+1$ or -1 signs associated to each simple root which does not pertain to the gravity line, and $i_1 i_2 \ldots i_t$ are the time indices. We shall verify that all these actions are equivalent and that their corresponding lagrangians are equal for all ξ (in the domain where the exponential parametrisation eq. (2.7) for each representation are both available). We show the equivalence by deriving the differential equations relating the fields parametrising the different coset representatives.

We write $S^{(t,s,\varepsilon)}_{\mathcal{G}^{++}_{(i_1 i_2 \ldots i_t)}}$ in terms of $dv^{(t,s,\varepsilon)}_{sym\,(i_1 i_2 \ldots i_t)}$ as

$$S^{(t,s,\varepsilon)}_{\mathcal{G}^{++}_{(i_1 i_2 \ldots i_t)}} = \int d\xi \frac{1}{n(\xi)} \left\langle \left(\frac{dv^{(t,s,\varepsilon)}_{sym\,(i_1 i_2 \ldots i_t)}(\xi)}{d\xi} \right)^2 \right\rangle, \tag{4.5}$$

where $dv^{(t,s,\varepsilon)}_{sym\,(i_1 i_2 \ldots i_t)}$ is computed as in eqs. (2.7) and (2.8), but using a Borel representative of $\mathcal{G}^{++}/K^{++}_{(-)}$, built from a gravity line endowed with the involution $\widetilde{\Omega}$ defining the signature (t, s, ε) with $\{i_1 i_2 \ldots i_t\}$ as time coordinates. This yields the same expressions as in eqs. (3.7), (3.8) and (3.9) but with a different space-time signature and a change of sign for all terms in levels generated by an odd number of simple roots not pertaining to the gravity line and carrying a negative contribution to ε.

We denote the Cartan generators $K^a{}_a$ and R as H^i and the tensor positive step operators as R^j. In terms of the differential forms $\{X_i, Y_j\}$ built from the Borel group fields , one has

$$dv^{(t,s,\varepsilon)}_{\text{sym}(i_1 i_2 \ldots i_t)} = \sum_i X_i H^i + \sum_j Y_j \left(R^j - \widetilde{\Omega} R^j \right). \tag{4.6}$$

On the other hand $S^{(t,s,\varepsilon)}_{\mathcal{G}^{++}_{(i_1 i_2 \ldots i_t)}}$ can be obtained by conjugation from $S_{\mathcal{G}_2^{++}}$ in the following way. Write $dv_{\mathrm{sym}(-)}$ as

$$dv_{\mathrm{sym}(-)} = \sum_i X'_i H'^{\,i} + \sum_j Y'_j (R'^{\,j} - \Omega_2 R'^{\,j})\,, \tag{4.7}$$

where $\{X'_i, Y'_j\}$ are the differential forms built from the Borel fields defined by a gravity line endowed with the involution Ω_2. Select generators $R''^{\,j}$ whose Weyl transform $R^i = \widetilde{U} R''^{\,j} \widetilde{U}^{-1}$, $\widetilde{U} \subset \mathcal{G}^{++}$, spans the positive step generators in the level expansion defining $dv^{(t,s,\varepsilon)}_{sym\,(i_1 i_2 \ldots i_t)}$ and maps the involution Ω_2 to $\widetilde{\Omega}$. We have

$$\widetilde{U}(R''^{\,j} - \Omega_2 R''^{\,j})\widetilde{U}^{-1} = \rho^j (R^j - \widetilde{\Omega} R^j)\,, \tag{4.8}$$

and ρ^j is a sign. Note that $R''^{\,j}$ need not be a positive step operator $R'^{\,j}$ in eq. (4.7) but one always has

$$(R''^{\,j} - \Omega_2 R''^{\,j}) = \lambda^j (R'^{\,j} - \Omega_2 R'^{\,j})\,, \tag{4.9}$$

where λ^j is a sign. Performing the conjugation on $dv_{\mathrm{sym}(-)}$ one gets

$$\widetilde{U} dv_{\mathrm{sym}(-)} \widetilde{U}^{-1} = \sum_i X'_i H^i + \sum_j Y'_j \lambda^j \rho^j (R^j - \widetilde{\Omega} R^j)\,. \tag{4.10}$$

Comparing eq. (4.6) with eq. (4.10) we see that the integrable differential equations relating the different Borel fields

$$\boxed{X'_i = X_i \;\;,\;\; Y'_j \lambda^j \rho^j = Y_j} \tag{4.11}$$

ensures that $dv^{(t,s,\varepsilon)}_{\mathrm{sym}\,(i_1 i_2 \ldots i_t)}$ and $dv_{\mathrm{sym}(-)}$ define the same lagrangian for all ξ and prove the equivalence of $S^{(t,s,\varepsilon)}_{\mathcal{G}^{++}_{(i_1 i_2 \cdots i_t)}}$ and $S_{\mathcal{G}^{++}_2}$. The resulting field transformations realise the Weyl transformation $\widetilde{U}$ in field space. We denote the set of equivalent actions by $S_{\mathcal{G}^{++}_B}$. The two distinct actions $S_{\mathcal{G}^{++}_B}$ and $S_{\mathcal{G}^{++}_C}$ are both contained as consistent distinct truncations of the unique action $S_{\mathcal{G}^{+++}}$, which encompasses different signatures and can be formulated, as was done for $S_{\mathcal{G}^{++}_B}$, as separate actions identified through field redefinitions.

We illustrate the field transformations obtained from eq. (4.11) by considering for $\mathcal{G}^{++} = E_8^{++}$ the equivalence of $S^{(1,9,+)}_{\mathcal{G}^{++}_{(9)}}$ and $S^{(2,8,-)}_{\mathcal{G}^{++}_{(10\,11)}}$, under the Weyl transformation $W_{\alpha_{11}}$ given in the first block of Table 2. First consider the non-trivial relation $Y'_j \lambda^j \rho^j = Y_j$ at the lowest level. Taking into account eq. (4.3) and eqs.(2.11), (2.12) we get the following non-linear relations (up to the sign $\lambda^j \rho^j$ not taken into account here)

$$\frac{1}{3!} dA'_{\mu\nu\rho} \left(e^{h'}\right)_8^{\;\mu} \left(e^{h'}\right)_{10}^{\;\nu} \left(e^{h'}\right)_{11}^{\;\rho} = -\left[e^{h}(de^{-h})\right]_8^{\;9}\,, \tag{4.12}$$

$$-\left[e^{h'}\left(de^{-h'}\right)\right]_8^{\;9} = \frac{1}{3!} dA_{\mu\nu\rho} \left(e^{h}\right)_8^{\;\mu} \left(e^{h}\right)_{10}^{\;\nu} \left(e^{h}\right)_{11}^{\;\rho}\,. \tag{4.13}$$

We now turn to the relation $X'_j = X_j$ between the Cartan generators. From $\sum_{i=2}^{11} X'_i H'^{\,i} \equiv -\sum_{a=2}^{11} dp'^a K^a_{\;a}$ $(p'^a = -h_a^{\;\prime a})$ one expresses the Weyl reflection $W_{\alpha_{11}}$ as [5]

$$K'^a_{\;\;a} = K^a_{\;a} \qquad a = 2, \ldots, 8$$
$$K'^a_{\;\;a} = K^a_{\;a} + \frac{1}{3}\left(K^2_{\;2} + \cdots + K^8_{\;8}\right) - \frac{2}{3}\left(K^9_{\;9} + K^{10}_{\;10} + K^{11}_{\;11}\right) \qquad a = 9, 10, 11$$

to recover the known result [5]

$$p'^a + \frac{1}{3}\left(p'^9 + p'^{10} + p'^{11}\right) = p^a \qquad a = 2, \dots, 8 \tag{4.14}$$

$$p'^a - \frac{2}{3}\left(p'^9 + p'^{10} + p'^{11}\right) = p^a \qquad a = 9, 10, 11\,. \tag{4.15}$$

We shall apply in the next section eqs. (4.12)–(4.15) to exact extremal brane solutions of $\mathcal{G}_B^{++}$.

5. Intersecting extremal branes in $\mathcal{G}^{++}$ theories

Exact solutions of $S_{\mathcal{G}^{+++}}$ giving the algebraic properties of intersecting brane configurations have been constructed in [19]. We show in this section that in $S_{\mathcal{G}_B^{++}}$, these coincide with the solutions of the Einstein and field equations of the maximally oxidised theories describing the intersecting branes smeared in all directions but one. ξ is identified to the non-compact coordinate.

We consider the level decomposition in $\mathcal{G}_B^{++}$ for a signature (t, s, ε) with $D - 1 = t + s$ connected by Weyl reflections to the 'phase' $(1, D-2, +)$ with time coordinate 2 of the $\mathcal{G}_B^{++}$ theory. This is the phase described by the fields in the action $S_{\mathcal{G}_2^{++}} \equiv S^{(1,9,+)}_{\mathcal{G}^{++}_{(2)}}$.

The equations of motion yielding the intersecting brane configuration solutions from $S_{\mathcal{G}^{+++}}$ in [19] can immediately be read in $S_{\mathcal{G}_2^{++}}$ provided the embedding relation eq. (3.3) is satisfied with $p_1 = \sum_{a=2}^{D} p_a$ labelling a space direction orthogonal to the branes. The solutions are unaltered for $a = 2, 3 \dots D$. As pointed out in [19], these solutions can be extended to exotic branes with t_A longitudinal timelike and s_A spacelike directions connected in $\mathcal{G}^{+++}$ by Weyl reflections. The results of section 4 yield therefore the intersecting brane configuration solutions, exotic or not, from the fields defining $S^{(t,s,\varepsilon)}_{\mathcal{G}^{++}_{(i_1 i_2 \dots i_t)}}$. For each of the $\mathcal{N}$ branes present in the configuration and characterised by $\tau_1 \dots \tau_{t_A}$ longitudinal timelike directions and $\lambda_1 \dots \lambda_{s_A}$ longitudinal spacelike directions, one has

$$A_{\tau_1 \dots \tau_{t_A} \lambda_1 \dots \lambda_{s_A}} = \epsilon_{\tau_1 \dots \tau_{t_A} \lambda_1 \dots \lambda_{s_A}} \left[\frac{2(D-2)}{\Delta_A}\right]^{1/2} H_A^{-1}(\xi) \qquad A = 1 \dots \mathcal{N}\,, \tag{5.1}$$

and

$$p^a = \sum_{A=1}^{\mathcal{N}} p_A^a = \sum_{A=1}^{\mathcal{N}} \frac{\eta_A^a}{\Delta_A} \ln H_A(\xi) \qquad a = 2, 3, \dots, D \tag{5.2}$$

$$\phi = \sum_{A=1}^{\mathcal{N}} \phi_A = \sum_{A=1}^{\mathcal{N}} \frac{D-2}{\Delta_A} \varepsilon_A a_A \ln H_A(\xi)\,. \tag{5.3}$$

Here $\eta_A^a = s_A + t_A$ or $-(D - 2 - s_A - t_A)$ depending on whether the direction a is perpendicular or parallel to the q_A-brane and $\Delta_A = (s_A + t_A)(D - 2 - s_A - t_A) + \frac{1}{2} a_A^2 (D-2)$. The factor ε_A is $+1$ for an electric brane and -1 for a magnetic one. Each of the branes in the configuration is thus described as electrically charged and is characterised by one positive

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harmonic function in ξ-space, namely one has

$$\frac{d^2 H_A(\xi)}{d\xi^2} = 0 \qquad A = 1 \dots \mathcal{N} \, . \tag{5.4}$$

From eq. (5.2), the embedding relation eq. (3.3) yields for the spatial direction 1 the result

$$p^1 = \sum_{A=1}^{\mathcal{N}} \frac{s_A + t_A}{\Delta_A} \ln H_A(\xi) \, , \tag{5.5}$$

identifying it to a direction transverse to all branes. The eqs. (5.2), (5.3) and (5.5) are solutions provided the generalised intersection rules [21]

$$\bar{s} + \bar{t} = \frac{(s_A + t_A)(s_B + t_B)}{D - 2} - \frac{1}{2} \varepsilon_A a_A \varepsilon_B a_B \, , \tag{5.6}$$

where $\bar{s}$ and $\bar{t}$ are the number of longitudinal space and time directions common to the two branes, are satisfied . The intersection rules eq. (5.6) can be expressed as an orthogonality condition between the real positive roots of $\mathcal{G}_B^{++}$ (and $\mathcal{G}^{+++}$) for all branes present in the configuration [19]. This orthogonality condition is in fact the input that permits the derivation of the exact solutions eqs. (5.1)–(5.4) by allowing a reduction of $S_{\mathcal{G}_B^{++}}$ to quadratic terms.

We may verify from the above equations that the lapse constraint in $S_{\mathcal{G}_B^{++}}$ is satisfied and takes the form

$$\sum_{a=2}^{D} (dp^a)^2 - \left(\sum_{a=2}^{D} dp^a \right)^2 + \frac{1}{2} (d\phi)^2 - \sum_{A=1}^{\mathcal{N}} \frac{D-2}{\Delta_A} (d \ln H_A)^2 = 0 \, , \tag{5.7}$$

where the differentials are taken in ξ-space. eq. (5.7) expresses the vanishing of the action for solutions involving fields associated to orthogonal roots. This condition is preserved under a Weyl reflection, whether or not the latter induces a signature change. It thus follows from the invariance of $S_{\mathcal{G}_B^{++}}$, as expressed by eq. (4.11), that the lapse equation is invariant under Weyl reflections. In addition the term $\Lambda = -\sum_{A=1}^{\mathcal{N}} (D - 2/\Delta_A)(d \ln H_A)^2$ is separately invariant because the other terms in eq. (5.7) are the quadratic form of $\mathcal{G}^{++}$ restricted to its Cartan subalgebra, which is Weyl invariant. This invariance relates different intersecting branes, Kaluza-Klein momenta and monopoles [10, 19] of different phases, conventional or exotic.

These exact solutions of $S_{\mathcal{G}_B^{++}}$ describe not only the algebraic properties of all corresponding solutions, exotic or not, of all the maximally oxidised theories and of their exotic counterparts related to them by 'dualities' encoded in the Weyl group of $\mathcal{G}_B^{++}$. They are now identical to the solutions of the Einstein and field equations describing these intersecting branes smeared in all directions but the spatial dimension 1. This follows on the one hand from the fact that the eq. (5.4) describes a harmonic function in one dimension as would be required by the latter solutions when ξ is identified with the single non-compact space coordinate x^1. On the other hand such identification is consistent with eq. (5.5) which shows that this non-compact direction is indeed transverse to all branes.

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It is instructive to illustrate by a specific example how the aforementioned solutions of the Einstein and field equations are transformed into themselves by the Weyl transformations ensuring the uniqueness of $S_{\mathcal{G}_B^{++}}$ for conventional and exotic solutions. We consider the mapping of the Kaluza-Klein momentum in 11-dimensional supergravity, smeared in all transverse directions but x^1 to the exotic smeared membrane with 2 longitudinal times. We take, as in table 2, the time to be 9 and the momentum in the direction 8. The metric is

$$ds^2 = -\widetilde{H}^{-1}(dx^9)^2 + \widetilde{H}\left[dx^8 + (\widetilde{H}^{-1} - 1)dx^9\right]^2 + (dx^1)^2 + \sum_{\mu=2\ldots7,10,11}(dx^\mu)^2\,, \tag{5.8}$$

where $\widetilde{H}(x^1)$ is a positive harmonic function in one dimension $\widetilde{H}(x^1) = A + Q|x^1|$. In the triangular gauge the relevant vielbein are given by

$$\begin{aligned} e_8{}^8 &= (2-\widetilde{H})^{-1/2} = H^{-1/2}\,,\\ e_9{}^9 &= (2-\widetilde{H})^{1/2} = H^{1/2}\,,\\ e_8{}^9 &= (2-\widetilde{H})^{-1/2} - (2-\widetilde{H})^{1/2} = H^{-1/2} - H^{1/2}\,. \end{aligned} \tag{5.9}$$

Here we have defined $H = 2 - \widetilde{H}$ and the range of x^1 is restricted to $H > 0$. These results, when expressed in terms of H, differs from the vielbein computed in of [10, appendix B1], where the time direction was 1, by a sign in the exponents of the diagonal time and space vielbein. This difference, which is a consequence of the triangular gauge when the time index is bigger than the space one, is crucial in eq. (5.13) below. The p'^a parametrising the Cartan subalgebra are from eq. (5.9)

$$p'^8 = -\frac{1}{2}\ln H\,, \qquad p'^9 = \frac{1}{2}\ln H\,, \tag{5.10}$$

while, using eq. (5.9), one may compute the non-zero non-Cartan field (see [10, appendix B1])

$$h_8{}'^9 = \ln H\,. \tag{5.11}$$

The vielbein equation eq. (5.9) also yields

$$-\,[e^{h'}(de^{-h'})]_8{}^9 = d\ln H\,. \tag{5.12}$$

We perform the Weyl reflection $W_{\alpha_{11}}$ using eq. (4.13)–(4.15) . For the Cartan we find

$$\begin{aligned} p^a &= \frac{1}{6}\ln H & a &= 2,3,4,5,6,7,9\\ p^a &= -\frac{1}{3}\ln H & a &= 8,10,11\,. \end{aligned} \tag{5.13}$$

For the non-Cartan fields we find from eqs. (4.13), (5.12) and (5.13), up to a sign,

$$dA_{8\,10\,11} = dH^{-1}\,. \tag{5.14}$$

Taking into account the relation eq. (5.5) this solution describes an exotic 'membrane' in the directions 8, 10 and 11, in a phase with signature $(2, 8, -)$ with time directions 10 and 11, characterised by the harmonic function $H = 2-\widetilde{H}$. Under this Weyl transformation the smeared KK-momentum of M-theory is thus mapped onto an exotic membrane with two longitudinal times of M^*-theory. The exotic membrane obtained is in perfect agreement with the T-duality interpretation as it can be checked by applying the Buscher transformations [28] on the KK-momentum solution in 10 dimensions obtained by reducing eq. (5.8) along the direction 11 and then uplifting back to M^*.

While eq. (5.13) illustrate the invariance of the quadratic form of $\mathcal{G}^{++}$ restricted to its Cartan subalgebra, eq. (5.14) confirms the invariance of the lapse constraint, yielding

$$\Lambda = -\frac{1}{2}(d\ln H)^2\,, \tag{5.15}$$

where, despite the fact that the membrane has two longitudinal times 10 and 11, a minus sign does arise in the quadratic lapse constraint from the negative kinetic term in the $(2,8,-)$ signature.

We point out that the Einstein solutions required the additional conditions

$$\Theta_E(-1)^{t_A+1} = 1 \qquad \Theta_E = (-1)^{t+1}\Theta_M\,, \tag{5.16}$$

where Θ_E and Θ_M are the signs of the kinetic terms for the electric and magnetic potentials in the exotic actions. These conditions are trivially satisfied in the conventional phase $S_{\mathcal{G}_2^{++}} \equiv S^{(1,9,+)}_{\mathcal{G}^{++}_{(2)}}$ where $\Theta_E = 1$. This phase is characterised by one time which is always longitudinal to the branes. Therefore they are automatically satisfied in all phases as the solutions in these phases can be obtained from Weyl transformations and hence do exist with Θ_E identified with the corresponding sign encoded in ε.

6. Perspectives

We have shown that all solutions of the cosmological and the brane overextended invariant actions $S_{\mathcal{G}_C^{++}}$ and $S_{\mathcal{G}_B^{++}}$ are solutions of the very-extended invariant actions $S_{\mathcal{G}^{+++}}$. The variable ξ used to parametrise the motion of the fields on the coset space $\mathcal{G}^{+++}/K^{+++}$ in the non-linear realisation $S_{\mathcal{G}^{+++}}$ of $\mathcal{G}^{+++}$ is then identified respectively to a time or to a space coordinate and a set of fields are consistently made to vanish. This suggests that the generators of $\mathcal{G}^{+++}$ contain a huge gauge redundancy. These generators are indeed associated to fields which must be interpreted as potentials rather than field strengths and equivalent descriptions may be possible with both space and time components of gauge fields. The conventional intersecting extremal branes, easily described by the $S_{\mathcal{G}_B^{++}}$, could perhaps also be obtained from $S_{\mathcal{G}_C^{++}}$ in a complicated way, while the latter action describes trivially the Kasner-like cosmological solutions.

Thus the fact that $\mathcal{G}^{+++}$ contains on equal footing time and space allows a selection of the best gauge potentials to describe a particular solution but this 'covariant' approach need not in principle comprehend a larger gauge invariant content than a non-covariant $\mathcal{G}^{++}$, as least as long as only conventional 'phases' of theories are considered. The existence of exotic solutions is a specific feature of $\mathcal{G}^{+++}$ which has no counterpart in the overextended action $S_{\mathcal{G}_C^{++}}$ and is a necessary concomitant of $\mathcal{G}^{+++}$-invariance and the tem-

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poral involution, as suggested in reference [17]. The only way exotic phases could decouple in very-extended algebra would be by restricting the signature to be euclidean, a rather problematic alternative.

An essential result of this work is the existence of solutions of $S_{\mathcal{G}^{+++}}$ and $S_{\mathcal{G}_B^{++}}$ which are identical to the space-time covariant solutions of intersecting extremal branes smeared in all directions but one. This was made possible by the interpretation of ξ as a spatial coordinate in the context of $S_{\mathcal{G}_B^{++}}$. In this way we do not have to restrict the interpretation of $S_{\mathcal{G}^{+++}}$ to algebraic considerations as in reference [10]. The reason of the importance of this result is that the exact solutions of the space-time covariant theories exist with the same algebraic structure but different, although simple, functional dependance in more uncompactified dimensions. Thus we have a laboratory to check whether or not the Kac-Moody theories can reach, at least in these simple cases, uncompactified gravity from the information contained in higher levels. If this turns out not to be the case, then the content of these approaches would probably only be a consistent dimensional reduction of the covariant space time theories down to one dimension through an infinity of equivalent fields. If uncompactified theories are reached, even at this elementary level, one has attained the first step of a formulation of gravity coupled to some matter fields, which is conceptually completely different from the Einstein approach, and which possibly includes new degrees of freedom which are hinted at by the string perturbative approaches. We hope to be able to provide the answer to this question in future work.

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